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Author(s)	Tanahashi, Kotaro; Yamagami, Shigeru
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Exponential Ordering on Bounded Self-Adjoint Operators

東北薬科大学　棚橋　浩太郎 (Kôtarô Tanahashi)
東北大学理学部　山上　滋 (Shigeru Yamagami)

[**概説**]

ヒルベルト空間上の有界自己共役作用素 A, B を考える。$O \leq A - B$ で $A - B$ が可逆のとき $B < A$ とかくことにする。次は藤井氏らによるおもしろい結果である。

Theorem 1 $A, B \in B(H)$ を可逆な正作用素とするとき次が成立する。

$$\log B < \log A \Longleftrightarrow \exists \alpha \in (0,1) : B^{\alpha} < A^{\alpha}$$

前回この研究集会で条件の不等号 $<$ を $\leq$ でおきかえられないことを紹介した。つまり次の不等式は無条件では成立しない。

$$\log B \leq \log A \Longleftrightarrow \exists \alpha \in (0,1) : B^{\alpha} \leq A^{\alpha}$$

そこで上の不等式が成立する条件を調べようとしたのがこの論文のきっかけである。昨年の研究集会ではヒルベルト空間の次元が 2 であれば次のように問題は解決していることを紹介した。

Theorem 2 A, B は *2* 行 *2* 列の可逆な正行列で $\log B \leq \log A$ であるが $\log B < \log A$ は満たさないとする。このとき次が成り立つ。

$$\exists \alpha > 0 : B^{\alpha} \leq A^{\alpha} \Longleftrightarrow AB = BA$$

しかしここで得られた $AB = BA$ という条件は強すぎてヒルベルト空間の次元が大きくなるとそのままでは成立しないことはすぐ分かる。

[**問**] A, B は 3 行 3 列の可逆な正行列で $\log B \leq \log A$ であるが $\log B < \log A$ は満たさないとする。このとき次が成り立つか。

$$\exists \alpha > 0 : B^{\alpha} \leq A^{\alpha} \Longleftrightarrow AB = BA$$

[**反例**] ($\Longleftarrow$) は自明である。($\Longrightarrow$) が成立しない例をあげよう。まず 2 行 2 列の行列で

$$\log B_2 < \log A_2, \quad A_2 B_2 \neq B_2 A_2$$

となるものをとる。このとき藤井氏らの結果から

$$\exists \alpha \in (0,1) : \ B_2^{\alpha} < A_2^{\alpha}$$

である。ここで

$$A = \begin{pmatrix} 1 & O \\ O & A_2 \end{pmatrix}, \quad B = \begin{pmatrix} 1 & O \\ O & B_2 \end{pmatrix}$$

とおくと

$$\log B = \begin{pmatrix} 0 & O \\ O & \log B_2 \end{pmatrix} \leq \log A = \begin{pmatrix} 0 & O \\ O & \log A_2 \end{pmatrix}$$
$$B^\alpha = \begin{pmatrix} 0 & O \\ O & B_2^\alpha \end{pmatrix} \leq A^\alpha = \begin{pmatrix} 0 & O \\ O & A_2^\alpha \end{pmatrix}$$

であるが $\log B < \log A$ は満たさない。しかし

$$AB = \begin{pmatrix} 1 & O \\ O & A_2 \end{pmatrix} \begin{pmatrix} 1 & O \\ O & B_2 \end{pmatrix} = \begin{pmatrix} 1 & O \\ O & A_2 B_2 \end{pmatrix}$$
$$\neq \begin{pmatrix} 1 & O \\ O & B_2 A_2 \end{pmatrix} = BA$$

である。

[証明終]

この例は $\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$ が A, B の共通の固有ベクトルになっているので、多分このような固有ベクトルがなければおなじ結論が得られると棚橋は考えていた。しかし、対称式を用いたトレースの計算は非常に複雑で直接的な計算は困難であったが、結局山上氏が次のように解決した。

Theorem 3 $A, B \in B(H)$ は自己共役で P を $\ker(B - A)$ への *orthogonal projection* とする。このとき

$$\exists \alpha \in (0,1): \ e^{\alpha A} \leq e^{\alpha B} \Longrightarrow A \leq B, PA = AP, PB = BP$$

が成立する。ヒルベルト空間 H が有限次元ならば逆も成立するが、無限次元ならば逆が成立しない例を作ることができる。

この結果を証明するのがこの論文の目的である。

1. Results

Let A, B be bounded selfadjoint operators in a Hilbert space $\mathcal{H}$. In [Ha], the notion of exponential ordering is introduced as the one defined by $e^A \leq e^B$. In this article, we consider an infinitesimal version of it: Consider the condition

$$e^{\kappa A} \leq e^{\kappa B} \qquad \text{for some } \kappa > 0,$$

which is equivalent to the following one by Löwner-Heinz' inequality: there is a positive real κ_0 such that

$$e^{\kappa A} \leq e^{\kappa B} \qquad \text{for all } 0 \leq \kappa \leq \kappa_0.$$

By the last expression, we see that the condition in fact defines an order relation in the set of bounded selfadjoint operators, which is weaker than the exponential ordering in [H] and will be referred to as **infinitesimal exponential ordering** in what follows.

By power series expansion in the exponential functions, the last condition is further equivalent to

$$B - A + \frac{\kappa}{2}(B^2 - A^2) + \frac{\kappa^2}{3!}(B^3 - A^3) + \cdots \geq 0 \quad \text{for sufficiently small } \kappa > 0,$$

which particularly implies the operator inequality $A \leq B$: the infinitesimal exponential ordering is finer than the ordinary ordering.

If $B - A$ is invertible, the converse implication is apparently true as remarked in [FJKT].

We here deal with the case when the kernel of $B - A$ is non-trivial and prove that the infinitesimal exponential inequality forces simultaneous decomposability of operators A, B with respect to the kernel projection P of $B - A$, i.e., $AP = PA$ and $PB = BP$.

When the Hilbert space $\mathcal{H}$ is finite-dimensional, this gives the following characterization of the infinitesimal exponential ordering: Let A and B be hermitian $n \times n$ matrices. The condition

$$e^{\kappa A} \leq e^{\kappa B} \qquad \text{for some } \kappa > 0$$

is then equivalent to require $PA = AP$, $PB = BP$ and $A \leq B$.

Since a generic operator inequality $A \leq B$ (under the assumption that $\ker(B - A) \neq 0$) does not satisfy the reducing property $PA = AP$, $PB = BP$, we have plenty of examples of operator inequality $A \leq B$ without satisfying the infinitesimal exponential order relation.

2. Proofs

With the notation in the previous section, express the operators A, B and P in a matrix form

$$A = \begin{pmatrix} a & b \\ b^* & c \end{pmatrix}, \qquad B = \begin{pmatrix} a' & b' \\ b'^* & c' \end{pmatrix}, \qquad P = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$$

with respect to the orthogonal decomposition $\mathcal{H} = (1 - P)\mathcal{H} + P\mathcal{H}$. By the choice of P, we have $(B - A)P = 0$, i.e., $c' = c$, $b' = b$ and

$$a' = a + h$$

with h an injective selfadjoint operator on $(1 - P)\mathcal{H}$.

Using these matrix expressions, we have

$$B-A+\frac{\kappa}{2}(B^2-A^2)+\frac{\kappa^2}{3!}(B^3-A^3)$$
$$=\begin{pmatrix} h+\frac{\kappa}{2}((a+h)^2-a^2)+\frac{\kappa^2}{6}((a+h)^3-a^3+bb^*h+hbb^*) & \frac{\kappa}{2}hb+\frac{\kappa^2}{6}((a+h)^2b-a^2b+hbc) \\ \frac{\kappa}{2}b^*h+\frac{\kappa^2}{6}(b^*(a+h)^2-b^*a^2+cb^*h) & \frac{\kappa^2}{6}b^*hb \end{pmatrix}$$

and hence the following expression for $(e^{\kappa B}-e^{\kappa A})/\kappa$:

$$\begin{pmatrix} h(k) & \frac{\kappa}{2}hb+\kappa^2 f(\kappa) \\ \frac{\kappa}{2}b^*h+\kappa^2 f(\kappa)^* & \frac{\kappa^2}{6}b^*hb+\kappa^3 r(\kappa) \end{pmatrix},$$

where $r(\kappa)$, $h(\kappa)$ and $f(\kappa)$ are operator-valued analytic functions of κ with $h(0)=h$.

Now the infinitesimal exponential order relation is equivalent to

$$C=\begin{pmatrix} h(k) & \frac{\kappa}{2}hb+\kappa^2 f(\kappa) \\ \frac{\kappa}{2}b^*h+\kappa^2 f(\kappa)^* & \frac{\kappa^2}{6}b^*hb+\kappa^3 r(\kappa) \end{pmatrix}\geq 0 \quad \text{for sufficiently small } \kappa\geq 0$$

and we need to prove $b=0$ from this condition.

By reducing the operator C to the subspace $(1-P)\mathcal{H}+\mathbb{C}\eta$ with η a normalized vector in $P\mathcal{H}$, we may assume that $P\mathcal{H}$ is one-dimensional, i.e.,

$$b,\ f(\kappa)\in(1-P)\mathcal{H}=\mathcal{L}(\mathbb{C}\eta,(1-P)\mathcal{H}) \qquad \text{and} \quad r(\kappa)\in\mathbb{C}.$$

We shall derive a contradiction if $b\neq 0$ by a series of arguments.

For each $\epsilon>0$, let e_ϵ be the spectral projection for h corresponding to the interval $[\epsilon,+\infty)$. By reducing the operator C by $\begin{pmatrix} e_\epsilon & 0 \\ 0 & 1\end{pmatrix}$, we obtain the operator inequality

$$C_\epsilon=\begin{pmatrix} h_\epsilon(k) & \frac{\kappa}{2}h_\epsilon b+\kappa^2 f_\epsilon(\kappa) \\ \frac{\kappa}{2}b^*h_\epsilon+\kappa^2 f(\kappa)^* & \frac{\kappa^2}{6}(b|hb)+\kappa^3 r(\kappa) \end{pmatrix}\geq 0 \quad \text{for sufficiently small } \kappa\geq 0,$$

where $h_\epsilon(\kappa)=e_\epsilon h(\kappa)e_\epsilon$ and $f_\epsilon(\kappa)=e_\epsilon f(\kappa)$. Note here that $h_\epsilon(\kappa)=h_\epsilon+O(\kappa)$ is invertible on $e_\epsilon\mathcal{H}$ for sufficiently small $\kappa\geq 0$ and $(b|hb)>0$ (h being injective and $b\neq 0$).

We now seek for a suitable eigenvector of C_ϵ for small $\kappa>0$ as an analytic perturbation of the selfadjoint operator h_ϵ.

To avoid notational complications, we first deal with the following problem: Let θ be a positive invertible operator on a Hilbert space $\mathcal{K}$, β be a non-trivial vector in $\mathcal{K}$ and γ be a real number. For a sufficiently small $\kappa>0$, consider the bounded self-adjoint operator

$$\begin{pmatrix} \theta & \kappa\beta \\ \kappa\beta^* & \kappa^2\gamma \end{pmatrix}$$

on the Hilbert space $\mathcal{K}\oplus\mathbb{C}$ and we seek for an eigenvector which converges to the vector $0\oplus 1\in\mathcal{K}\oplus\mathbb{C}$ projectively as κ goes to 0.

The eigenrelation

$$\begin{pmatrix} \theta & \kappa\beta \\ \kappa\beta^* & \kappa^2\gamma \end{pmatrix}\begin{pmatrix}\xi \\ y\end{pmatrix}=\lambda\begin{pmatrix}\xi \\ y\end{pmatrix}$$

with $\xi\in\mathcal{K}$ and $y\in\mathbb{C}$ is equivalent to the equations

$$(\lambda-\kappa^2\gamma)y=\kappa(\beta|\xi)$$
$$((\lambda-\kappa^2\gamma)\theta+\kappa^2\beta\beta^*)\xi=\lambda(\lambda-\kappa^2\gamma)\xi.$$

(Note that, for $\lambda=\kappa^2\gamma$, the above equations imply $(\xi|\theta\xi)=\kappa^2\gamma(\xi|\xi)$ and hence (θ being invertible) $\xi=0$ together with $y=0$ for small $\kappa>0$, i.e., if λ is an eigenvalue, $\lambda\neq\kappa^2\gamma$.)

We here assume that the vector ξ has the expression

$$\xi=\sum_{n\geq 0}x_n\theta^{-n-1}\beta \qquad \text{with } x_n\in\mathbb{C}.$$

The eigenrelation for ξ is then satisfied if

$$(\lambda-\kappa^2\gamma)x_0\beta+\kappa^2(\beta|\xi)\beta+(\lambda-\kappa^2\gamma)\sum_{n\geq 1}x_n\theta^{-n}\beta=\lambda(\lambda-\kappa^2\gamma)\sum_{n\geq 1}x_{n-1}\theta^{-n}\beta.$$

Since the family $\{\theta^{-n}\beta\}_{n\geq 0}$ is linearly independent for a generic θ, we try to solve the equation by comparing the coefficients of $\theta^{-n}\beta$: the equation for ξ is (formally) satisfied if

$$(\lambda-\kappa^2\gamma)x_0+\kappa^2\sum_{n\geq 0}x_n(\beta|\theta^{-n-1}\beta)=0$$
$$(\lambda-\kappa^2\gamma)x_n=\lambda(\lambda-\kappa^2\gamma)x_{n-1}\qquad (n\geq 1).$$

Since $\lambda-\kappa^2\gamma\neq 0$, the above requirements are reduced to

$$\lambda-\kappa^2\gamma+\kappa^2\sum_{n\geq 0}(\beta|\theta^{-n-1}\beta)\lambda^n=0$$

with

$$\xi=x_0\sum_{n\geq 0}\lambda^n\theta^{-n-1}\beta.$$

We now rewrite the equation for λ into the form

$$\kappa^2=\frac{\lambda}{\gamma-\sum_{n\geq 0}(\beta|\theta^{-n-1}\beta)\lambda^n}.$$

Since $(\beta|\theta^{-n-1}\beta)\leq\|\beta\|^2\|\theta^{-1}\|^{n+1}$, the formal power series

$$w=\frac{z}{\gamma-\sum_{n\geq 0}(\beta|\theta^{-n-1}\beta)z^n}$$

of z is convergent in a neighborhood of $0\in\mathbb{C}$ and, for $\gamma\neq(\beta|\theta^{-1}\beta)$, it is univalent near $z=0$ and z can be expressed as an analytic function of w. Thus, if $\gamma\neq(\beta|\theta^{-1}\beta)$, we have an absolutely convergent power series expression

$$\lambda(\kappa^2)=\sum_{n\geq 1}\lambda_n\kappa^2,\quad \lambda_1=\gamma-(\beta|\theta^{-1}\beta)$$

for sufficiently small $\kappa>0$ so that it satisfies the equation for λ.

Now the formal expression

$$\xi=x_0\sum_{n\geq 0}\lambda(\kappa^2)^n\theta^{-n-1}\beta$$

turns out to be absolutely convergent for small $\kappa>0$ as $\lambda(\kappa^2)=O(\kappa^2)$ and $\|\theta^{-n-1}\beta\|\leq\|\theta^{-1}\|^{n+1}\|\beta\|$.

As a conclusion, if $\gamma\neq(\beta|\theta^{-1}\beta)$, then the selfadjoint operator

$$\begin{pmatrix}\theta & \kappa\beta\\ \kappa\beta^* & \kappa^2\gamma\end{pmatrix}$$

has an eigenvector of eigenvalue $\kappa^2\mu(\kappa^2)$ with the analytic function $\mu(\kappa^2)$ of κ^2 determined by the equation

$$\gamma-(\beta|\theta^{-1}\beta)-\sum_{n\geq 1}(\beta|\theta^{-n-1}\beta)(\kappa^2\mu)^n=\mu.$$

To apply these analyses to the case $\mathcal{K}=e_\epsilon\mathcal{H}$, $\theta=h_\epsilon(\kappa)$, $\beta=h_\epsilon b/2+\kappa f_\epsilon(\kappa)$ and $\gamma=(b|hb)/6+\kappa r(\kappa)$, we need to make a closer look into the behavior of μ when ϵ and κ converge to 0 in a suitable way.

To simplify the notation, we set

$$\rho_0=(\beta|\theta^{-1}\beta)-\gamma\qquad\text{and}\qquad \rho_n=(\beta|\theta^{-n-1}\beta)\quad\text{for } n\geq 1.$$

Then the defining equation of μ takes the form

$$\sum_{n\geq 0}\rho_n\kappa^{2n}\mu^n=-\mu.$$

Lemma 1. *Express $h(\kappa)=h+\kappa g(\kappa)$ with $g(\kappa)$ an operator-valued analytic function of κ and set*

$$F=\sup\{\|f(\kappa)\|;0\leq\kappa\leq 1\},\quad G=\sup\{\|g(\kappa)\|;0\leq\kappa\leq 1\},\quad R=\sup\{|r(\kappa)|;0\leq\kappa\leq 1\}.$$

Then we have

$$\|h_\epsilon(\kappa)^{-1}\|\leq\frac{1}{\epsilon-G\kappa},$$

$$\left|\rho_0 - \frac{1}{12}(b|hb)\right| \le R\kappa + \frac{1}{4}\|b\|^2\epsilon + \left(\frac{1}{4}G\|b\|^2 + F\|hb\| + F^2\right)\frac{\kappa}{\epsilon - G\kappa}$$

and, for $n \ge 1$,

$$\rho_n \le \frac{\|hb\|^2/4 + F\|hb\| + F^2}{(\epsilon - G\kappa)^{n+1}}$$

whenever $G\kappa < \epsilon$, $\kappa \le 1$ and $\epsilon \le 1$.

Proof. By Neumann series expansion,

$$h_\epsilon(\kappa)^{-1} = h_\epsilon^{-1}(1 + \kappa g_\epsilon(\kappa)h_\epsilon^{-1})^{-1} = h_\epsilon^{-1}\sum_{n\ge 0}(-\kappa g_\epsilon(\kappa)h_\epsilon^{-1})^n,$$

which gives the estimate

$$\|h_\epsilon(\kappa)^{-1}\| \le \|h_\epsilon^{-1}\|\sum_{n\ge 0}(\kappa G\|h_\epsilon^{-1}\|)^n \le \epsilon^{-1}\sum_{n\ge 0}\left(\frac{G\kappa}{\epsilon}\right)^n = \frac{1}{\epsilon - G\kappa},$$

proving the first inequality.

To obtain the second inequality, we estimate $(\beta|\theta^{-1}\beta) - (b|h_\epsilon b)/4$ as follows:

$$\begin{aligned}
&|(\beta|\theta^{-1}\beta) - \frac{1}{4}(b|h_\epsilon b)| \\
&= \left|(\frac{1}{2}h_\epsilon b + \kappa f_\epsilon(\kappa)|h_\epsilon(\kappa)^{-1}(\frac{1}{2}h_\epsilon b + \kappa f_\epsilon(\kappa))) - \frac{1}{4}(b|h_\epsilon b)\right| \\
&= \left|\frac{1}{4}(b|h_\epsilon(h_\epsilon(\kappa)^{-1} - h_\epsilon^{-1})h_\epsilon b) + \kappa\mathcal{R}e(h_\epsilon b|h_\epsilon(\kappa)^{-1}f_\epsilon(\kappa)) + \kappa^2(f_\epsilon(\kappa)|h_\epsilon(\kappa)^{-1}f_\epsilon(\kappa))\right| \\
&\le \frac{1}{4}\left|(b|\sum_{n\ge 1}(-\kappa g_\epsilon(\kappa)h_\epsilon^{-1})^n h_\epsilon b)\right| + \|h_\epsilon b\|\|f_\epsilon(\kappa)\|\frac{\kappa}{\epsilon - G\kappa} + \|f_\epsilon(\kappa)\|^2\frac{\kappa^2}{\epsilon - G\kappa} \\
&\le \frac{1}{4}\sum_{n\ge 1}\|b\|^2\frac{(G\kappa)^n}{\epsilon^{n-1}} + F\|hb\|\frac{\kappa}{\epsilon - G\kappa} + F^2\frac{\kappa^2}{\epsilon - G\kappa} \\
&= \left(\frac{\|b\|^2}{4}G\kappa\epsilon + F\|hb\| + F^2\kappa\right)\frac{\kappa}{\epsilon - G\kappa}.
\end{aligned}$$

The third inequality is of a similar taste and comes from

$$\rho_n \le \frac{1}{4}(h_\epsilon b|h_\epsilon(\kappa)^{-n-1}h_\epsilon b) + \kappa\left|(h_\epsilon b|h_\epsilon(\kappa)^{-n-1}f_\epsilon(\kappa))\right| + \kappa^2(f_\epsilon(\kappa)|h_\epsilon(\kappa)^{-n-1}f_\epsilon(\kappa))$$

together with the first inequality. □

Lemma 2. *We can find a positive real $\delta = \delta(h, b, F, G) < 1$ such that*

$$\rho_1 \ge \frac{1}{8}\frac{\|hb\|^2}{\|h\|^2}$$

whenever $0 < \epsilon \le \delta$, $0 < \kappa \le \delta$ and $0 < \kappa/(\epsilon - G\kappa)^2 \le \delta$.

Proof. This follows from

$$\begin{aligned}
(\beta|\theta^{-2}\beta) &= \frac{1}{4}(b|h_\epsilon h_\epsilon(\kappa)^{-2}h_\epsilon b) + \kappa\mathcal{R}e(h_\epsilon b|h_\epsilon(\kappa)^{-2}f_\epsilon(\kappa)) + \kappa^2(f_\epsilon(\kappa)|h_\epsilon(\kappa)^{-2}f_\epsilon(\kappa)) \\
&\ge \frac{1}{4}\frac{\|h_\epsilon b\|^2}{\|h(\kappa)\|^2} - F\|hb\|\frac{\kappa}{(\epsilon - G\kappa)^2} - F^2\frac{\kappa^2}{(\epsilon - G\kappa)^2}.
\end{aligned}$$

Here we used the operator inequality $\|h(\kappa)\|^{-1}1 \le h_\epsilon(\kappa)^{-1}$ in the second line. □

We now combine all these to get

Lemma 3. *Let $\kappa = \epsilon^4$. Then there are positive reals $\delta = \delta(h, b, F, G) < 1$ and $M = M(h, b, F, G) > 1$ such that, if $\epsilon \leq \delta$, then $\rho_0 \leq M$, $\rho_1 \geq M^{-1}$ and*

$$\frac{\rho_n}{\rho_1}\kappa^{n-1}M^{n-1} \leq 1$$

for $n \geq 1$.

Proof. By the previous lemmas, we can find $M > 1$ such that $\rho_0 \leq M$ and $\rho_1 \geq M^{-1}$ for sufficiently small $\epsilon > 0$. The third inequality is trivial if $n = 1$. For $n \geq 2$, the estimate

$$\begin{aligned}\frac{\rho_n}{\rho_1}\kappa^{n-1}M^{n-1} &\leq 2\left(||h||^2 + 4\frac{F||h||^2}{||hb||} + 4\frac{F^2||h||^2}{||hb||^2}\right)\frac{(M\kappa)^{n-1}}{(\epsilon - G\kappa)^{n+1}} \\ &= 2\left(||h||^2 + 4\frac{F||h||^2}{||hb||} + 4\frac{F^2||h||^2}{||hb||^2}\right)\frac{M^{n-1}\epsilon^{3n-5}}{(1 - G\epsilon^3)^{n+1}}\end{aligned}$$

shows that the left-hand side converges to 0 uniformly in $n \geq 2$ when ϵ goes to 0. □

From the inequality

$$\begin{aligned}\rho_n\kappa^{2n}M^n \leq M\rho_1\kappa^{n+1} &\leq M\left(\frac{||hb||^2}{4} + F||hb|| + F^2\right)\frac{\kappa^{n+1}}{(\epsilon - G\kappa)^2} \\ &= M\left(\frac{||hb||^2}{4} + F||hb|| + F^2\right)\frac{\epsilon^{4n+2}}{(1 - G\epsilon^3)^2}\end{aligned}$$

for $n \geq 1$, the power series

$$\sum_{n\geq 0}\rho_n\kappa^{2n}t^n$$

defines a real analytic function $\varphi(t)$ for $|t| < M$. From the inequality,

$$\begin{aligned}\varphi'(t) &= \sum_{n\geq 1} n\rho_n\kappa^{2n}t^{n-1} \\ &\geq \rho_1\kappa^2 - \sum_{n\geq 2} n\rho_n\kappa^{2n}M^{n-1} \\ &\geq \rho_1\kappa^2 - \sum_{n\geq 2}\rho_1 n\kappa^{n+1} \\ &= \rho_1\kappa^2\left(2 - \frac{1}{(1-\kappa)^2}\right),\end{aligned}$$

the function $\varphi(t)$ is strictly increasing in $-M < t < M$ for sufficiently small $\epsilon > 0$, whence the equation $\varphi(t) = -t$ has a unique solution $\mu = \mu_\epsilon$ in the interval $(-M, 0)$.

The inequality

$$|\mu + \rho_0| \leq \sum_{n\geq 1}\rho_n\kappa^{2n}|\mu|^n \leq \sum_{n\geq 1}\rho_n\kappa^{2n}M^n \leq M\rho_1\sum_{n\geq 1}\kappa^{n+1} = M\frac{\rho_1\kappa^2}{1-\kappa}$$

then shows that

$$\lim_{\epsilon\to+0}\mu_\epsilon = -\lim_{\epsilon\to+0}\rho_0 = -\frac{1}{12}(b|hb) < 0$$

and therefore

$$|\lambda| = |\kappa^2\mu| \leq (b|hb)\kappa^2$$

for sufficiently small $k = \epsilon^4$.

Now the estimate

$$\begin{aligned}|\lambda|^n||\theta^{-n-1}\beta|| &= \frac{1}{2}|\lambda|^n||h_\epsilon(\kappa)^{-n-1}(h_\epsilon b + 2\kappa f_\epsilon(\kappa))|| \\ &\leq \left(\frac{1}{2}||hb|| + F\right)\frac{(b|hb)^n\kappa^{2n}}{(\epsilon - G\kappa)^{n+1}}\end{aligned}$$

shows that the summation

$$\xi = \sum_{n\geq 0}\lambda^n\theta^{-n-1}\beta$$

in $e_\epsilon\mathcal{H}$ is absolutely convergent for sufficiently small $\kappa = \epsilon^4$ and the previous arguments on analytic perturbations prove that ξ gives rise to an eigenvector of C_ϵ of eigenvalue $\kappa^2\mu_\epsilon$, which contradicts with the assumption $C_\epsilon \geq 0$ because $\mu_\epsilon < 0$ for sufficiently small $\epsilon > 0$ for $\kappa = \epsilon^4$.

3. Examples

For a pair of bounded self-adjoint operators (A, B) satisfying $A \leq B$, we set

$$\kappa(A,B) = \sup\{\kappa \geq 0; e^{\kappa A} \leq e^{\kappa B}\},$$

which has the following obvious properties:

$$\begin{cases} \kappa(A + c1, B + c1) = \kappa(A,B) & \text{if } c \text{ is a real number.} \\ \kappa(cA, cB) = \frac{1}{c}\kappa(A,B) & \text{if } c \text{ is a positive real.} \\ \kappa(UAU^*, UBU^*) = \kappa(A,B) & \text{if } U \text{ is a unitary operator.} \end{cases}$$

When A and B are 2×2 hermitian matrices, after the composition of these three operations, the pair (A, B) takes the form

$$A = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \quad B = \lambda \begin{pmatrix} \cos^2\theta & \cos\theta\sin\theta \\ \cos\theta\sin\theta & \sin^2\theta \end{pmatrix} + \mu \begin{pmatrix} \sin^2\theta & -\cos\theta\sin\theta \\ -\cos\theta\sin\theta & \cos^2\theta \end{pmatrix}$$

with λ, μ reals except for the trivial case that A is a scalar matrix (use the angle representation of two projections).

The condition of majorization $A \leq B$ is then equivalent to

$$0 \leq \lambda\sin^2\theta + \mu\cos^2\theta \leq \lambda\mu,$$

which particularly implies $\lambda \geq 0$, $\mu \geq 0$.

Now the following is easy to check:

Proposition 4. *Assume that* $\cos\theta\sin\theta \neq 0$. *Then, for* $\lambda \geq 0$, $\mu \geq 0$, *we have*

$$\begin{cases} \kappa(A,B) = +\infty & \text{if and only if } \lambda \geq 1 \text{ and } \mu \geq 1. \\ 0 < \kappa(A,B) < +\infty & \text{if and only if } (\lambda - 1)(\mu - 1) < 0,\ \lambda\sin^2\theta + \mu\cos^2\theta < \lambda\mu. \\ \kappa(A,B) = 0 & \text{if and only if } (\lambda - 1)(\mu - 1) < 0,\ \lambda\sin^2\theta + \mu\cos^2\theta = \lambda\mu. \end{cases}$$

For example, choose $\sin\theta = \cos\theta = 1/\sqrt{2}$ and

$$\lambda_n^{-1} = \frac{1}{2} - \frac{1}{n}, \qquad \mu_n^{-1} = \frac{3}{2} - \frac{1}{n}$$

for $n \geq 3$. Then

$$B_n = \frac{2n}{(n-2)(3n-2)} \begin{pmatrix} 2n-2 & n \\ n & 2n-2 \end{pmatrix}$$

majorates A with the limit

$$B = \lim_{n\to\infty} B_n = \frac{2}{3}\begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}$$

and these satisfy $\kappa(A, B_n) > 0$, $\kappa(A,B) = 0$.

Now we are ready to construct an example of bounded self-adjoint operators $A' \leq B'$ with no infinitesimal exponential order relation and having the trivial kernel for the difference $B' - A'$: Let $A' \leq B'$ be defined on the Hilbert space $\bigoplus_{n\geq 3} \mathbb{C}^2$ by

$$A' = \bigoplus_{n\geq 3} \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \qquad B' = \bigoplus_{n\geq 3} B_n.$$

Then clearly $\ker(B' - A') = \{0\}$. If $\kappa = \kappa(A', B') = \inf\{\kappa(A, B_n); n \geq 3\} > 0$, $e^{\kappa A} \leq e^{\kappa B_n}$ for any $n \geq 3$ and therefore, by taking the limit $n \to \infty$, $e^{\kappa A} \leq e^{\kappa B}$, which is impossible because $\kappa(A,B) = 0$.

References

[1] T. Ando, *On some operator inequalities,* Math. Ann., 279(1987), 157–159.

[2] M. Fujii, J.F. Jiang, E. Kamei and K. Tanahashi *A characterization of chaotic order and a problem,* J. Inequalities and Applications, to appear.

[3] M. Fujii and E. Kamei, *Furuta's inequality and a generalization of Ando's Theorem,* Proc. Amer. Math. Soc., 115(1992), 409–413.

[4] T. Furuta, *$A \geq B \geq 0$ assures $(B^r A^p B^r)^{1/q} \geq B^{(p+2r)/q}$ for $r \geq 0$, $p \geq 0$, $q \geq 1$ with $(1+2r)q \geq p+2r$,* Proc. Amer. Math. Soc., 101(1987), 85–88.

[5] F. Hansen, *Selfadjoint means and operator monotone functions,* Math. Ann., 256(1981), 29–35.

[6] E. Heinz, *Beiträge zur Störungstheorie der Spektralzerlegung,* Math. Ann., 123(1951), 415–438.

[7] T. Kato, *Perturbation Theory for Linear Operators,* Springer-Verlag, 1976.

[8] K. Löwner, *Über monotone Matrixfunktionen,* Math. Z., 38(1934), 177–216.

Department of Mathematics, Tohoku College of Pharmacy, Sendai, 981, Japan

Department of Mathematics, Tohoku University, Sendai, 980, Japan